Hw 9 Suggested Exercises

1. $X(u,v) = (f(v)\cos u,\ f(v)\sin u,\ g(v))$

$X_u = (-f(v)\sin u,\ f(v)\cos u,\ 0)$

$X_v = (f'(v)\cos u,\ f'(v)\sin u,\ g'(v))$

$$(g_{ij}) = \begin{pmatrix} f^2(v) & 0 \\ 0 & f'(v)^2 + g'(v)^2 \end{pmatrix}$$

$g_{11,2} = 2ff'$, $g_{22,2} = 2(f'f'' + g'g'')$. Otherwise , $g_{ij,k} = 0$

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} (g_{il,j} + g_{jl,i} - g_{ij,l})$$

$$\Gamma^1_{11} = \Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = 0 \ , \ \Gamma^1_{12} = \frac{f'}{f} \ , \ \Gamma^2_{11} = \frac{ff'}{f'^2 + g'^2} \ , \ \Gamma^2_{22} = \frac{f'f'' + g'g''}{f'^2 + g'^2}$$

2. Let S be the saddle surface $\{z = x^2 - y^2\}$

We need to show Gauss curvature of $S < 0$

Since local isometry preserves Gauss curvature and Gauss curvature of sphere or cylinder ≥ 0 ,

S is not locally isometric to sphere or cylinder

$\vdash$: Gauss curvature of $S < 0$

Let $X(u,v) = (u, v, u^2 - v^2)$ $(u,v) \in \mathbb{R}^2$ be parametrization of S

$X_u = (1, 0, 2u)$, $X_v = (0, 1, -2v)$

$$N = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{1}{\sqrt{1+4u^2+4v^2}} (-2u, 2v, 1)$$

$X_{uu} = (0,0,2)$, $X_{uv} = (0,0,0)$, $X_{vv} = (0,0,-2)$

$$(g_{ij}) = \begin{pmatrix} 1+4u^2 & -4uv \\ -4uv & 1+4v^2 \end{pmatrix} \qquad (h_{ij}) = \frac{1}{\sqrt{1+4u^2+4v^2}} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$K = \frac{\det(h_{ij})}{\det(g_{ij})} = -\frac{4}{(1+4u^2+4v^2)^2} < 0$$, K: Gauss curvature

Hw 10 Suggested Exercises

1. Let S be the connected surface, $p \in S$

First, we claim that $\forall\, v \in T_pS$, v is eigenvector of dN_p

~~The~~ This implies $\kappa_1(p) = \kappa_2(p)$ where κ_1, κ_2 are principal curvature

Hence S would be totally umbilic

So S must be contained in plane or a sphere

$\vdash$: $\forall\, v \in T_pS$, v is eigenvector of dN_p

Let $v \in T_pS$, consider geodesic $\alpha : (-\varepsilon, \varepsilon) \to S$ s.t. $\alpha(0) = p$, $\alpha'(0) = \frac{v}{\|v\|}$

Since α is geodesic, $|\alpha'|$ is constant and hence α is p.b.a.l.

Let N be normal of S

N_α be normal of α, T_α be tangent of α, B_α be binormal of α

As $\nabla_{\alpha'}\alpha' = 0$, we have $(\alpha''(t))^T = 0$, so $\alpha''(t) \,/\!/\, N$

Also $\alpha''(t) = kN_\alpha$, $N_\alpha \,/\!/\, N \implies N_\alpha = \pm N$

Therefore,

$$dN_p(v) = \left.\frac{dN}{dt}\right|_{t=0} = \pm \left.\frac{dN_\alpha}{dt}\right|_{t=0} = \pm(-kT_\alpha - \tau B_\alpha) = \pm kT_\alpha = \pm kv$$

(as α is contained in plane, $\tau = 0$)

2. $D_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k + A_{ij}N$ (Gauss equation)

$D_{\partial_i}N = -g^{jk}A_{ij}\partial_k$ (Weingarten equation)

2a. $\vdash$: $\partial_\ell\partial_i N = -[\partial_\ell(g^{kj}A_{ij}) + g^{pj}\Gamma^k_{\ell p}A_{ij}]\partial_k - (g^{pj}A_{\ell p}A_{ij})N$

$$\partial_\ell\partial_i N = \partial_\ell(-g^{jk}A_{ij}\partial_k)$$
$$= [\partial_\ell(-g^{jk}A_{ij})]\partial_k - g^{jk}A_{ij}D_{\partial_\ell}\partial_k$$

$$= -[\partial_\ell (g^{jk} A_{ij})]\partial_k - g^{jk} A_{ij} (\Gamma^p_{\ell k} \partial_p + A_{\ell k} N)$$

$$= -[\partial_\ell (g^{jk} A_{ij})]\partial_k - g^{jk} A_{ij} \Gamma^p_{\ell k} \partial_p - g^{jk} A_{ij} A_{\ell k} N$$

(Interchange the index p, k for last two term)

$$= -[\partial_\ell (g^{jk} A_{ij})]\partial_k - g^{jp} A_{ij} \Gamma^k_{\ell p} \partial_k - g^{jp} A_{ij} A_{\ell p} N$$

$$= -[\partial_\ell (g^{kj} A_{ij}) + g^{pj} \Gamma^k_{\ell p} A_{ij}]\partial_k - (g^{pj} A_{\ell p} A_{ij}) N$$

2b. $\sum_p g_{ip} g^{pq} = \delta_i^q$, $\delta_i^j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

$$(\partial_\ell g_{ip}) g^{pq} + g_{ip} (\partial_\ell g^{pq}) = 0$$

$$g_{ip} (\partial_\ell g^{pq}) = -g^{pq} (\partial_\ell g_{ip})$$

$$g^{ki} g_{ip} (\partial_\ell g^{pq}) = -g^{ki} g^{pq} (\partial_\ell g_{ip})$$

Sum over i, $\delta_p^k (\partial_\ell g^{pq}) = -g^{ki} g^{pq} (\partial_\ell g_{ip})$

$$\partial_\ell g^{kq} = -g^{ki} g^{pq} (\partial_\ell g_{ip})$$

2c. $\vdash$: $\partial_\ell g^{kq} + g^{pq} \Gamma^k_{\ell p} = -g^{kj} \Gamma^q_{j\ell}$

$$\partial_\ell g^{kq} + g^{pq} \Gamma^k_{\ell p} = -g^{qi} g^{kp} \partial_\ell g_{pi} + g^{pq} (\tfrac{1}{2} g^{ki}) (g_{\ell i,p} + g_{pi,\ell} - g_{\ell p,i})$$

$$= -g^{qi} g^{kp} \partial_\ell g_{pi} + \tfrac{1}{2} g^{qi} g^{kp} (g_{\ell p,i} + g_{pi,\ell} - g_{\ell i,p})$$

(Interchange the index i and p)

$$= \tfrac{1}{2} g^{qi} g^{kp} (g_{\ell p,i} - g_{pi,\ell} - g_{\ell i,p})$$

$$= -g^{kp} \Gamma^q_{\ell p}$$

2d. Tangential component of $\partial_l \partial_i N$: $-[\partial_l (g^{kj} A_{ij}) + g^{pj}\Gamma^k_{lp} A_{ij}]\, \partial_k$

$$= -[(\partial_l g^{kj}) A_{ij} + g^{pj}\Gamma^k_{lp} A_{ij} + g^{kj}(\partial_l A_{ij})]\, \partial_k$$

$$= -[(\partial_l g^{kj} + g^{pj}\Gamma^k_{lp}) A_{ij} + g^{kj}(\partial_l A_{ij})]\, \partial_k$$

$$= -[-g^{kp}\Gamma^j_{pl} A_{ij} + g^{kj}(\partial_l A_{ij})]\, \partial_k$$

$$= \left(g^{kp}\Gamma^j_{pl} A_{ij} - g^{kj}(\partial_l A_{ij})\right) \partial_k$$

Similarly, tangential component of $\partial_i \partial_l N$: $\left(g^{kp}\Gamma^j_{pi} A_{lj} - g^{kj}(\partial_i A_{lj})\right) \partial_k$

Since $\partial_l \partial_i N = \partial_i \partial_l N$,

$$g^{kp}\Gamma^j_{pl} A_{ij} - g^{kj}(\partial_l A_{ij}) = g^{kp}\Gamma^j_{pi} A_{lj} - g^{kj}(\partial_i A_{lj})$$

$$g_{sk} g^{kp}\Gamma^j_{pl} A_{ij} - g_{sk} g^{kj}(\partial_l A_{ij}) = g_{sk} g^{kp}\Gamma^j_{pi} A_{lj} - g_{sk} g^{kj}(\partial_i A_{lj})$$

Sum over k,

$$\delta^p_s \Gamma^j_{pl} A_{ij} - \delta^j_s(\partial_l A_{ij}) = \delta^p_s \Gamma^j_{pi} A_{lj} - \delta^j_s(\partial_i A_{lj})$$

$$\Gamma^j_{sl} A_{ij} - \partial_l A_{is} = \Gamma^j_{si} A_{lj} - \partial_i A_{ls} \quad \text{(Codazzi equation)}$$